Alternative algebras with hyperbolic unit loops*

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Abstract

Let $\mathbb{K}$ be a quadratic extensions of the field of rational numbers. We investigate the structure of an alternative finite dimensional $\mathbb{K}$-algebra $\mathfrak{A}$ subject to the condition that for some $\mathbb{Z}$-order $\Gamma \subset \mathfrak{A}$, the loop of units of $\mathcal{U}(\Gamma)$ does not contain a free abelian subgroup of rank two. As a result, we give a complete classification of the finite and infinite RA-loops L for which $\mathbb{K}L$ has this property. In particular if $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$, we show that L is the Cayley loop and $d \equiv 7 \pmod 8$ is positive and square free. The complete classification for group rings is still an open problem.

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1 Introduction

Group rings $\mathbb{Z}G$ whose unit groups $\mathcal{U}(\mathbb{Z}G)$ are hyperbolic were characterized in [8] in case G is polycyclic-by-finite. A similar question was considered for RG, R being the ring of algebraic integers of $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ and G a finite group (see [9]). In [6, 7], these results were extended to associative algebras $\mathcal{A}$ of finite dimension over the rational numbers containing a $\mathbb{Z}$-order $\Gamma \subset \mathcal{A}$ whose unit group $\mathcal{U}(\Gamma)$ is hyperbolic. An algebra $\mathcal{A}$ with this property is said to have the hyperbolic property. Using these general results, the finite semigroups S and the field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ such that $\mathbb{K}S$ has the hyperbolic property were classified.

In this paper we study the same problem in the context of non-associative algebras, in special those which are loop algebras. A loop L is a nonempty set with a closed binary operation $\cdot$ relative to which there is a two-sided identity element and such that the right and left translation maps $R_x(g) := g \cdot x$ and $L_x(g) := x \cdot g$ are bijections. L is said to be hyperbolic if it does not contain a free abelian subgroup of rank two. This definition is an extension of the notion of hyperbolic group defined by Gromov [5] via the Flat Plane Theorem [2, Corollary $III.\Gamma.3.10.(2)$].

Here we characterize the RA-loops L and the rings of integers $\mathfrak{o}_{\mathbb{K}}$ of $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ such that $\mathcal{U}(\mathfrak{o}_{\mathbb{K}}L)$ has the hyperbolic property.

In section 2, we fix notation and give definitions. For fields $\mathbb{K} \subset \mathbb{Q}(\sqrt{-d})$, we focus on the alternative algebras of finite dimension over $\mathbb{K}$ with the hyperbolic property and prove that the Cayley-Dickson algebra over $\mathbb{K}$, where $d \equiv 7 \pmod 8$ is a positive integer, has the hyperbolic property. In section 3, we prove a structure theorem for finite dimensional alternative algebras with the hyperbolic property. As a consequence we obtain that, for these algebras, the radical associates with the whole algebra. In the last section we present the main result , giving a full classification of

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those RA-loops L whose unit loop $\mathcal{U}(\mathbb{Z}L)$ is hyperbolic and extend this also to $\mathfrak{o}_{\mathbb{K}}L$. It is important to notice that this problem is not yet completely settled for groups.

2 The Hyperbolic Property

$\mathcal{D} = \{d \in \mathbb{Z} \setminus \{-1, 0\} : c^2 \nmid d,\ c \in \mathbb{Z},\ c^2 \neq 1\}$ denotes the set of square free integers. For a field $\mathbb{K}$ let $\mathbf{H}(\mathbb{K}) = (\frac{\alpha,\beta}{\mathbb{K}}),\ \alpha, \beta \in \mathbb{K}$ be the generalized quaternion algebra over $\mathbb{K}$, i.e., $\mathbf{H}(\mathbb{K}) = \mathbb{K}[i, j : i^2 = -\alpha, j^2 = -\beta, ij = -ji =: k]$. The set $\{1, i, j, k\}$ is a $\mathbb{K}$-basis of $\mathbf{H}(\mathbb{K})$. Such an algebra is a totally definite quaternion algebra, which we will denote $\mathbb{K}(-\alpha, -\beta)$, if $\mathbb{K}$ is totally real and α, β are totally positive. The map $\eta : \mathbf{H}(\mathbb{K}) \longrightarrow \mathbb{K}$, $\eta(x = x_1 + x_i i + x_j j + x_k k) = x_1^2 - \alpha x_i^2 - \beta x_j^2 + \alpha\beta x_k^2$ is called norm map.

Denoting by $[x, y, z] \doteq (xy)z - x(yz)$, recall that a ring A is alternative if $[x, x, y] = [y, x, x] = 0$, for every $x, y \in A$. Let $\mathfrak{A}$ be a finite dimension alternative $\mathbb{Q}$-algebra. By [10, Theorem 3.18], $\mathfrak{A} \cong \mathfrak{S} \oplus \mathfrak{R}$, where $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$ and $\mathfrak{S} \cong \mathfrak{A}/\mathfrak{R}$ is semi-simple.

Definition 2.1. Let $\mathbb{K}$ be a field, of characteristic zero, and let $\mathfrak{A}$ be an alternative finite dimension $\mathbb{K}$-algebra. We say $\mathfrak{A}$ has the *hyperbolic property* if there exists a $\mathbb{Z}$-order $\Gamma \subset \mathfrak{A}$ whose unit loop $\mathcal{U}(\Gamma)$ is a hyperbolic loop.

For an associative finite dimensional $\mathbb{Q}$-algebra this property was coined the *hyperbolic property* (see [6]). We will use this name also in the non-associative setting.

Proposition 2.2. *Let $\mathfrak{A}$ be an alternative finite dimension $\mathbb{Q}$-algebra such that $\mathfrak{A} \cong \mathfrak{S} \oplus \mathfrak{R}$, with $\mathfrak{R}$ being the radical of $\mathfrak{A}$. If $\mathfrak{A}$ has the hyperbolic property, then $\mathfrak{R}$ is 2-nilpotent. Furthermore, there exists $j_0 \in \mathfrak{R}$ such that $j_0^2 = 0$ and $\mathfrak{R} \cong \langle j_0 \rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$-linear span of j_0 over $\mathbb{Q}$.*

Proof. Let $x, y \in \mathfrak{A}$, by Artin's Theorem the subalgebra generated by x, y is an associative algebra. Thus the result follows from [6, Lemma 3.2 and Corollary 3.3] $\square$

Definition 2.3. Let $\mathcal{A}$ be an algebra over a field $\mathbb{K}$. An involution is a $\mathbb{K}$-linear map $\star : \mathcal{A} \to \mathcal{A}$ $a^\star := \overline{a}$ satisfying $(a \cdot b)^\star = b^\star \cdot a^\star$ and $(a^\star)^\star = a$. The map $n : \mathcal{A} \to \mathbb{K}$, $n(a) := a \cdot \overline{a}$, is called a norm on $\mathcal{A}$.

We recall the Cayley-Dickson duplication process: Let $\mathcal{A}$ be a given $\mathbb{K}$-algebra, with $char(\mathbb{K}) \neq 2$, $\alpha \in \mathbb{K}$ and x an indeterminate over $\mathcal{A}$, such that $x^2 = \alpha$. The composition algebra $\mathfrak{A} = (\mathcal{A}, \alpha)$, is the algebra whose elements are of the form $a + bx$, where $a, b \in \mathcal{A}$, with operations defined as follows:

$(+)$ $(a_1 + b_1 x) + (a_2 + b_2 x) := (a_1 + a_2) + (b_1 + b_2)x$;

$(\cdot)$ $(a_1 + b_1 x) \cdot (a_2 + b_2 x) := (a_1 a_2 + \alpha \overline{b}_2 b_1) + (b_2 a_1 + b_1 \overline{a}_2)x$

On $\mathfrak{A}$, we have a natural involution defined by $\overline{a + bx} := \overline{a} - bx$.

The algebra $\mathfrak{A} := (\mathbb{K}, \alpha, \beta, \gamma)$ is the decomposition algebra $(\mathcal{A}, \gamma)$, where $\mathcal{A} = (\mathbb{K}, \alpha, \beta)$ is the generalized quaternion algebra $H(\mathbb{K}) := (\frac{\alpha,\beta}{K})$, with $\alpha, \beta \in \mathbb{K}$. Writing $\mathfrak{A} = \{u + vz : u, v \in \mathcal{A}\}$ we have that $\mathcal{B} = \{1, x, y, xy\} \cup \{z, xz, yz, (xy)z\}$ is a $\mathbb{K}$-basis of $\mathfrak{A}$ with $x^2 = \alpha, y^2 = \beta, z^2 = \gamma$. Moreover $n(a_1 + a_x x + a_y y + a_{xy} xy + a_z z + a_{xz} xz + a_{yz} yz + a_{(xyz)}(xy)z) = a_1^2 - a_x^2\alpha - a_y^2\beta + a_{xy}^2\alpha\beta - a_z^2\gamma + a_{xz}^2\alpha\gamma + a_{yz}^2\beta\gamma - a_{(xyz)}^2\alpha\beta\gamma$ is a norm.

Lemma 2.4. *Let $\mathcal{A}$ be the Cayley-Dickson algebra $(\mathbb{K}, -1, -1, -1)$, $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$, $d \in \mathcal{D}$ be a quadratic rational extension and $\mathfrak{o}_{\mathbb{K}}$ the ring of algebraic integers of $\mathbb{K}$. The algebra $\mathcal{A}$ has the hyperbolic property if, and only if, d is positive and $d \equiv 7 \pmod 8$.*

Proof.

Since the quaternion algebra $\mathbf{H}(\mathbb{K}) \cong (\mathbb{K}, -1, -1)$ over $\mathbb{K}$ is a subalgebra of $\mathcal{A}$, if $\mathcal{A}$ has the hyperbolic property, then for all $\mathbb{Z}$-order $\Gamma \subset \mathbf{H}(\mathbb{K})$, the group $\mathcal{U}(\Gamma)$ does not contain a free abelian subgroup of rank two, in particular $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(H(\mathfrak{o}_{\mathbb{K}}))$. Therefore, by [9, Theorem 4.7], $d \equiv 7 \pmod 8$ and $d > 1$.

Conversely, since $d \equiv 7 \pmod 8$ and d is positive we claim that $\mathcal{A}$ is a division algebra. In fact, suppose $\mathcal{A}$ splits. By [3, Theorem $I.3.4$], $\mathcal{A}$ splits if, and only if, the equation $x^2 + y^2 = -z^2$ has non trivial solution in $\mathbb{K}$ and this yields also that $(\mathbb{K}, -1, -1)$ splits, but $(\mathbb{K}, -1, -1) \cong \mathbf{H}(\mathbb{K})$, and the quaternion algebra $\mathbf{H}(\mathbb{K})$ over $\mathbb{K}$ is a division ring if, and only if, d is positive and $d \equiv 7 \pmod 8$, contradicting the fact that $(\mathbb{K}, -1, -1)$ splits. Suppose there exists a $\mathbb{Z}$-order $\Gamma \subset \mathcal{A}$ with $\mathbb{Z}^2 \hookrightarrow \Gamma$. Hence there exists $u, v \in \mathcal{U}(\Gamma)$ such that $\langle u, v \rangle \cong \mathbb{Z}^2$. Let $\mathbb{L} := \mathbb{K}[u, v]$ be the ring generated by $\{u, v\}$ over $\mathbb{K}$, since $[\mathcal{A} : \mathbb{Q}] = 16$ and $\mathcal{A}$ is diassociative $\mathbb{L} = \mathbb{K}(u, v)$ is a field and there exists $\beta \in \mathcal{A}$ such that $\mathbb{L} = \mathbb{K}(\beta)$. Obviously β is not central in $\mathcal{A}$, because $\mathbb{K} \subsetneq \mathbb{L}$, hence there exist $\gamma \in \mathcal{A}, \gamma\beta \neq \beta\gamma$, therefore the algebra $H := \mathbb{L}(\gamma) = K(\beta, \gamma)$ is a division ring. We claim that $H \cong \mathbf{H}(\mathbb{K})$ the quaternion algebra over $\mathbb{K}$; in fact the algebra $F = \mathbb{Q}(\beta, \gamma)$ is a division ring. Computing the degree $[H : \mathbb{Q}] = [H : \mathbb{K}][\mathbb{K} : \mathbb{Q}] = [H : \mathbb{K}] \cdot 2$, but $\mathcal{A} \supsetneq H$ because H is associative, hence $[H : \mathbb{Q}]$ is at most 8. Since F is a division ring $[F : \mathbb{Q}] \geq 4$. Since $u, v \notin \mathbb{K}$ thus $\beta \notin \mathbb{K}$, also $\mathbb{K}$ is the center of H hence $\gamma \notin \mathbb{K}$; clearly $[H : \mathbb{K}] = [F : \mathbb{Q}]$ thus $[F : \mathbb{Q}] = 4$ and F is a quaternion algebra over $\mathbb{Q}$ therefore $H \cong \mathbf{H}(\mathbb{K})$. This last condition shows that there exists $\Gamma' \subset H$ and $\mathbb{Z}^2 \hookrightarrow \mathcal{U}(\Gamma')$, but by [9, Theorem 4.7] H has the hyperbolic property, a contradiction. □

Definition 2.5 (Alternative Totally Definite Octonion Algebra)**.** An alternative division algebra $\mathfrak{A}$ whose center is a field $\mathbb{K}$ is called an *alternative totally definite octonion algebra* if $\mathbb{K}$ is totally real and $\mathcal{B} = \{1, x, y, xy\} \cup \{z, xz, yz, (xy)z\}$ is a $\mathbb{K}$-basis of $\mathfrak{A}$, with $x^2 = -\alpha$, $y^2 = -\beta$, $z^2 = -\gamma$ and $\alpha, \beta, \gamma \in \mathbb{K}$ all totally positive elements. In this case we write $\mathfrak{A} := \mathbb{K}(-\alpha, -\beta, -\gamma)$.

One should compare our definition with those in [3] and [11, Chapter 3, Section 21]. An example of such an algebra is $(\mathbb{Q}, -1, -1, -1)$, which is non-split.

The alternative totally definite octonion algebra $\mathfrak{A} := \mathbb{K}(-\alpha, -\beta, -\gamma)$ is non-split: since α, β, γ are totally positive and $\mathbb{K}$ is totally real, the equation $x^2 + \alpha y^2 + \beta z^2 + \alpha\beta w^2 + \gamma t^2 = 0$, $x, y, z \in \mathbb{K}$, has only the trivial solution. Thus, by [3, Theorem 3.4], $\mathfrak{A}$ is non-split.

Next we give a characterization of the alternative totally definite octonion algebras which is a naturally extension of [11, Lemma 21.3].

Theorem 2.6. *Let $\mathfrak{A}$ be a non commutative alternative division algebra, finite dimensional over its center $\mathbb{K}$. Suppose that $\mathbb{K}$ is a number field, $\mathfrak{o}_{\mathbb{K}}$ its ring of algebraic integers and $\mathfrak{O}$ a maximal order in $\mathfrak{A}$. Then the following are equivalent.*

1. *$SL_1(\mathfrak{O})$, the loop of units in $\mathfrak{O}$ having reduced norm 1, is finite;*
2. *$|\mathcal{U}(\mathfrak{O}) : \mathcal{U}(\mathfrak{o}_{\mathbb{K}})| < \infty$;*
3. *$\mathfrak{A}$ is an alternative totally definite octonion algebra.*

Proof. (1) $\Rightarrow$ (2): The reduced norm η_1 induces a map: $\varphi : \mathcal{U}(\mathfrak{O})/\mathcal{U}(\mathfrak{o}_{\mathbb{K}}) \longrightarrow \mathcal{U}(\mathfrak{o}_{\mathbb{K}})/(\mathcal{U}(\mathfrak{o}_{\mathbb{K}}))^2$. If $x\mathcal{U}(\mathfrak{o}_{\mathbb{K}}) \in ker(\varphi)$, then $\varphi(x\mathcal{U}(\mathfrak{o}_{\mathbb{K}})) = \eta_1(x)(\mathcal{U}(\mathfrak{o}_{\mathbb{K}}))^2 \in (\mathcal{U}(\mathfrak{o}_{\mathbb{K}}))^2$ and $\eta_1(x) := \lambda^2 \in (\mathcal{U}(\mathfrak{o}_{\mathbb{K}}))^2 \cap \mathbb{K}$. Define $z := \lambda^{-1}x \in \mathcal{U}(\mathfrak{O})$, clearly $\eta_1(z) = 1$ thus $\lambda^{-1}x \in SL_1(\mathfrak{O})$ and the kernel of φ is $SL_1(\mathfrak{O})\mathcal{U}(\mathfrak{o}_{\mathbb{K}})/\mathcal{U}(\mathfrak{o}_{\mathbb{K}})$. Since we are dealing with finitely generated abelian groups we have that $|\mathcal{U}(\mathfrak{O}) : \mathcal{U}(\mathfrak{o}_{\mathbb{K}})| < \infty$.

(2) $\Rightarrow$ (3): Let D be a division ring which is maximal in $\mathfrak{A}$. Since $|\mathcal{U}(\mathfrak{O}) : \mathcal{U}(\mathfrak{o}_{\mathbb{K}})| < \infty$, if $\Gamma \subset D$ is a $\mathbb{Z}$-order, then $|\mathcal{U}(\Gamma) : \mathcal{U}(\mathfrak{o}_{\mathbb{K}})| < \infty$. By [11, Lemma 21.3], D is a totally definite quaternion algebra and $\mathbb{K}$ is totally real . Let $x \in \mathfrak{A}$ and $B = (K, x)$. If D is a maximal subalgebra of $\mathfrak{A}$ with $B \subset D$, then B must be a quadratic field. Let $D_0 < \mathfrak{A}$ be a maximal subalgebra and $x_0 \in \mathfrak{A} \setminus D_0$. Then $\mathfrak{A} = (D_0, x_0)$ and we may suppose that $x_0^2 \in \mathbb{K}$. Since D_0 is a totally definite quaternion algebra, then $D_0 = (\frac{a_0,b_0}{\mathbb{K}})$. Thus $E := (\mathbb{K}, -a_0, -x_0)$, with $x_0,\ a_0 \in \mathbb{K}$, is a totally definite quaternion algebra, hence $\mathfrak{A} = E + Ej_0$, with $j_0^2 = -b_0 \in \mathbb{K}$, is a totally definite octonion algebra.

(3) $\Rightarrow$ (1) is a consequence of [11, Lemma 21.3]. □

Let $\mathcal{P}$ be a a theoretical group property. Recall that a group G is virtually $\mathcal{P}$ if it has a subgroup of finite index with the property $\mathcal{P}$. Also, if G and H are commensurable groups, then there exists subgroups $K \leq G$ and $L \leq H$, both of finite index, which are isomorphic.

Lemma 2.7. *Let $\mathfrak{A} = \mathbb{K}(-\alpha, -\beta, -\gamma)$ be an alternative totally definite octonion algebra over a number field $\mathbb{K}$, and $\mathfrak{O} \subset \mathfrak{A}$ a maximal $\mathbb{Z}$-order of $\mathfrak{A}$. The unit loop $\mathcal{U}(\mathfrak{O})$ is a hyperbolic loop if and only if $\mathbb{K} \in \{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}) : -d > 1$ a square free integer$\}$.*

Proof. If $\mathcal{U}(\mathfrak{O})$ is a hyperbolic loop, then $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(\mathfrak{O})$. Suppose $\mathcal{U}(\mathfrak{O})$ is finite. Let $O \subset \mathbb{K}$ be an order of $\mathbb{K}$, $\mathcal{U}(O) \subset \mathcal{U}(\mathfrak{O})$ is a finite subgroup. Since $\mathbb{K}$ is totally real, by Dirichlet's Unit Theorem, $\mathbb{K} = \mathbb{Q}$. Suppose $\mathcal{U}(\mathfrak{O})$ is infinite. For $a, b \in \{\alpha, \beta, \gamma\}, a \neq b$ we have, since $\mathbb{K}$ is totally real, that the algebra $\mathcal{A} = \mathbb{K}(-a, -b)$ is a totally definite quaternion algebra. Let $\mathfrak{o}_{\mathcal{A}} \subset \mathcal{A}$ be a $\mathbb{Z}$-order, then either $|\mathcal{U}(\mathfrak{o}_{\mathcal{A}})| = \infty$ or $\mathbb{K} = \mathbb{Q}$. In either case, since $\mathcal{U}(\mathfrak{o}_{\mathcal{A}})$ and $\mathcal{U}(\mathfrak{O})$ are commensurable, $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(\mathfrak{o}_{\mathcal{A}})$. By [11, item (b) of Lemma 21.3], $|\mathcal{U}(\mathfrak{o}_{\mathcal{A}}) : \mathcal{U}(O)|$ is finite, therefore $\mathcal{U}(O)$ is virtually cyclic, thus $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ where the integer $-d > 1$.

Conversely, if $\mathbb{K} = \mathbb{Q}$, then $\mathcal{U}(\mathfrak{o}_{\mathbb{K}})$ is finite and, by the item (2) of the last theorem, $\mathcal{U}(\mathfrak{O})$ is finite. If $\mathbb{K} = \mathbb{Q}(\sqrt{-d}), -d > 1$, then, by Dirichlet Unit Theorem, $\mathcal{U}(\mathfrak{o}_{\mathbb{K}}) \cong \mathbb{Z}$ and item (2) of last theorem yields that $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(\mathfrak{O})$. Hence $\mathcal{U}(\mathfrak{O})$ is a hyperbolic loop. □

3 A Structure Theorem

In this section we prove a structure theorem for alternative algebra with the hyperbolic property. This is not only a crucial step in the classification of the RA-loops whose integral loop algebra has the hyperbolic property but it should be also of independent interest.

Theorem 3.1. *Let $\mathfrak{A}$ be an alternative algebra of finite dimension over $\mathbb{Q}$, $\mathcal{A}_i$ a simple epimorphic image of $\mathfrak{A}$, and $\Gamma_i \subset \mathcal{A}_i$ a $\mathbb{Z}$-order. Then*

1. *The algebra $\mathfrak{A}$ has the hyperbolic property, is semi-simple and without non-zero nilpotent elements if, and only if,*
$$\mathfrak{A} = \oplus\mathcal{A}_i,$$
and for at most one index i_0 the loop $\mathcal{U}(\Gamma_{i_0})$ is infinite and hyperbolic.

2. *The algebra $\mathfrak{A}$ has the hyperbolic property, is semi-simple with non-zero nilpotent element if, and only if,*
$$\mathcal{A} = (\oplus \mathcal{A}_i) \oplus M_2(\mathbb{Q})$$
and for all i the loop $\mathcal{U}(\Gamma_i)$ is finite.

3. *The algebra $\mathfrak{A}$ has the hyperbolic property and is non-semi-simple with central radical J if, and only if,*
$$\mathfrak{A} = (\oplus \mathcal{A}_i) \oplus J, dim_{\mathbb{Q}}(J) = 1$$
and for all i the loop $\mathcal{U}(\Gamma_i)$ is finite.

4. *The algebra $\mathfrak{A}$ has the hyperbolic property and is non-semi-simple with non-central radical if, and only if,*
$$\mathfrak{A} = (\oplus \mathcal{A}_i) \oplus T_2(\mathbb{Q}).$$

For each of the items $(1), (2), (3)$ *and* (4)*, at least one index j is such that $\mathcal{A}_j$ is an alternative totally definite octonion algebra over $\mathbb{Q}$. The components $\mathcal{A}_i$ are one of the following algebras.*

i. $\mathbb{Q}$

ii. *a quadratic imaginary extension of $\mathbb{Q}$*

iii. *a totally definite quaternion algebra over $\mathbb{Q}$*

iv. *an alternative totally definite octonion algebra over $\mathbb{Q}$ whose center, except for the index i_0 of item* (1)*, has finitely many units.*

Furthermore, in the decompositions of $(1), (2), (3)$ *and* (4)*, every simple epimorphic image of $\mathfrak{A}$ in the direct sum is an ideal of $\mathfrak{A}$. It follows also that the radical associates with the whole algebra.*

Now, we will work toward a proof of this theorem, proving it at the end of this section.

From now on, $\mathbb{K}$ denotes the quadratic extension $\mathbb{Q}\sqrt{-d}$, where $d \in \mathcal{D}$. Let F be a field of characteristic $char(F) \neq 2$. The Cayley-Dickson algebras we refer here are 8-dimensional algebras constructed in the previous section. We shall start to look at the algebra $(\mathbb{K}, -1, -1, -1)$ and to one of its $\mathbb{Z}$-order $(\mathfrak{o}_{\mathbb{K}}, -1, -1, -1)$.

A Cayley-Dickson algebra $\mathcal{A}$ is a simple non-associative alternative ring which may have zero-divisors. If $\mathcal{A}$ does not split, then it is said to be a division ring.

If R is a ring, then we denote by $\mathfrak{Z}(R)$, the Zorn vector matrix algebra over R. This is a split simple alternative algebra.

Clearly if $\{\theta_1, \theta_2\}$ is a $\mathbb{Z}$-independent set of commuting nilpotent elements then $\langle 1+\theta_1, 1+\theta_2\rangle \cong \mathbb{Z}^2$. We will use this in our next result.

Proposition 3.2. *The Zorn vector matrix algebra over $\mathbb{Q}$, $\mathfrak{Z}(\mathbb{Q})$, does not have the hyperbolic property.*

Proof. $\Lambda = \mathfrak{Z}(\mathbb{Z})$ is a $\mathbb{Z}$-order of $\mathfrak{Z}(\mathbb{Q})$ and if $e_1 := (1, 0, 0)$ and $e_2 := (0, 1, 0)$ then
$$\theta_1 := \begin{pmatrix} 0 & e_1 \\ (0) & 0 \end{pmatrix} \quad and \quad \theta_2 := \begin{pmatrix} 0 & e_2 \\ (0) & 0 \end{pmatrix}.$$
are 2-nilpotent element which are $\mathbb{Z}$-independent. The result now follows. □

For a commutative and associative unital ring R, a loop L and a group G, the loop ring RL and the group ring RG have been objects of intensive research (see [3, chap. III], [11, chap. 1], [12]).

We will concentrate on RA-loops, i.e., a loop L whose loop algebra RL over some commutative, associative and unitary ring R of characteristic not equal to 2 is alternative, but not associative (see [3]).

RA-loops are Moufang Loops, i.e., loops satisfying any one of the following Moufang identities:

1. $((xy)x)z = x(y(xz))$ the left Moufang identity;
2. $((xy)z)y = x(y(zy))$ the right Moufang identity;
3. $(xy)(zx) = (x(yz))x$ the middle Moufang identity;

The following duplication process of a group results in Moufang loops. It turns out that all RA-loops are obtained in this way. Let G be a nonabelian group, $g_0 \in \mathcal{Z}(G)$ be a central element, $\star : G \to G$ be an involution such that $g_0^\star = g_0$ and $gg^\star \in \mathcal{Z}(G)$, for all $g \in G$, and u be an indeterminate. The set $L = G\dot{\cup}Gu =: M(G, \star, g_0)$, with the operations

1. $(g)(hu) = (hg)u$;
2. $(gu)h = (gh^\star)u$;
3. $(gu)(hu) = g_0h^\star g$,

is a Moufang Loop (see [3]).

A Hamiltonian loop is a non-associative loop L whose subloops are all normal. A theorem of Norton gives a complete characterization of these loops ([3, Theorem II.8]). In what follows E stands for an elementary abelian 2-group and Q_8 stands for the quaternion group of order 8.

Proposition 3.3. *Let G be a Hamiltonian 2-group and $L = M(G, *, g_0)$. Then L is an RA-loop which is a Hamiltonian 2-loop and $\mathcal{U}_1(\mathbb{Z}L) = L$.*

Proof. This is well know and [4, Theorem 2.3] is a good reference. □

Lemma 3.4. *Let L be a finite RA-loop. If the algebra $\mathbb{Q}L$ has nonzero nilpotent elements, then $\mathbb{Q}L$ has a simple epimorphic image which is isomorphic to Zorn's matrix algebra over $\mathbb{Q}$.*

Proof. This is well known and follows from [3, Corollary VI.4.3] and [3, Corollary VI.4.8]. □

The last fact that we need is that alternative algebras are diassociative (see [9]).

Proof. (of Theorem 3.1): Suppose $\mathfrak{A}$ has the hyperbolic property, is semi-simple without non-trivial nilpotent elements. Let $\Gamma \subset \mathfrak{A}$ be a $\mathbb{Z}$-order. We may suppose that $\Gamma = \oplus\Gamma_i$, where each $\Gamma_i \subset \mathcal{A}_i$ is a $\mathbb{Z}$-order, and so $\mathcal{U}(\Gamma) = \prod \mathcal{U}(\Gamma_i)$. If $\mathcal{U}(\Gamma)$ is finite, then is each group $\mathcal{U}(\Gamma_i)$, thus by Theorem 2.6 and [11, Lemma 21.2], the components $\mathcal{A}_i$ are determined. If $\mathcal{U}(\Gamma)$ is infinite, then there exists a unique index i_0 such that $\mathcal{U}(\Gamma_{i_0})$ is infinite, otherwise $\mathcal{U}(\Gamma)$ is not hyperbolic. By the hypothesis we have that $\mathcal{U}(\Gamma_{i_0})$ is a hyperbolic loop. Conversely, let $\Gamma \subset \mathcal{A}$ be a $\mathbb{Z}$-order and let $\mathfrak{o}_{\mathcal{A}_i}$ be the ring of algebraic integers of $\mathcal{A}_i$. By hypothesis, $\Gamma_0 = \oplus\mathfrak{o}_{\mathcal{A}_i}$ is such that $\mathcal{U}(\Gamma_0)$ is hyperbolic. Since $\mathcal{U}(\Gamma)$ and $\mathcal{U}(\Gamma_0)$ are commensurable, we have that $\mathcal{U}(\Gamma)$ is a hyperbolic loop.

To prove item (2), we suppose that $\mathfrak{A}$ has the hyperbolic property, is semi-simple with non-trivial nilpotent elements. If $\mathcal{A}_i$ is non-associative, then, by Proposition 3.2 each $\mathcal{A}_i$ is a division algebra. Since the algebra $\mathfrak{A}$ has a non-zero nilpotent element, clearly there exists exactly one component

$\mathcal{A}_j \cong M_2(\mathbb{Q})$. To see this, by Lemma 3.4 and Proposition 3.2, if $\mathcal{A}_i$ contains a nilpotent element, then $\mathcal{A}_i$ is associative and by [6, Theorem 3.1], this component is $M_2(\mathbb{Q})$. Since $\mathcal{U}(M_2(\mathbb{Q})) \cong GL_2(\mathbb{Z})$ contains a copy of $\mathbb{Z}$, it follows that each other component $\mathcal{A}_i, i \neq j$ is as predicted. Conversely, observe that $GL_2(\mathbb{Z})$ is an infinite hyperbolic group. It now easily follows that $\mathcal{A}$, whose components $\mathcal{A}_i$ are prescribed, has the hyperbolic property.

Item (3): Proposition 2.2 assures that the radical J has dimension 1 over $\mathbb{Q}$, $J = \langle j_0 \rangle_{\mathbb{Q}}$. Since J is central and $\langle 1 + j_0 \rangle \cong \mathbb{Z}$ it clearly follows that for each $\mathcal{A}_i$ any $\mathbb{Z}$-order $\Gamma_i \subset \mathcal{A}_i$. We must have that $\mathcal{U}(\Gamma_i)$ is finite and $\mathcal{A}_i$ is as described. The converse is also obvious.

Item (4): By [10, Theorem 3.18], $\mathfrak{A} \cong \mathfrak{S} \oplus \mathfrak{R}$ whose $\mathfrak{S} \cong \oplus_{i=1}^{N} \mathcal{A}_i$. Assume $\mathfrak{A}$ has the hyperbolic property, then $\mathfrak{R} = \mathbb{Q}j_0$, where $j_0^2 = 0$. Let $\{e_i / e_i \in \mathcal{A}_i\}$ be the set of primitive central idempotents of $\mathfrak{S}$. For each idempotent e_i, $e_i \cdot j_0 \in \mathfrak{R}$ and hence, $e_i \cdot j_0 = \lambda_i j_0$. Since $e_i = e_i^2$ and $\mathcal{A}_i$ is diassociative, we have $e_i \cdot (e_i \cdot j_0) = e_i \cdot j_0 = \lambda_i e_i$, also $e_i \cdot (e_i \cdot j_0) = e_i \cdot (\lambda_i j_0) = \lambda_i^2 e_i$, thus $\lambda_i^2 = \lambda_i$. Since $\mathcal{A}$ is unitary, $1 = e_1 + \cdots + e_N$, we have that $j_0 = j_0(\lambda_1 + \cdots + \lambda_N)$ and hence $\sum \lambda_i = 1$. So, there exists a unique index I such that $e_I \cdot j_0 = j_0$. Similarly, there exists a unique index J, such that $j_0 \cdot e_J = j_0$. Reordering indexes, we have that $e_1 \cdot j_0 = j_0 \cdot e_N = j_0$. Let M be the annihilator of j_0 in $\mathcal{A}_1$. It is easily seen that M is closed under addition, multiplication and left and right multiplication by any element of $\mathcal{A}_1$. To see this use the fact that, the radical being one dimensional, $a \cdot j_0 = \lambda j_0$, for some $\lambda \in \mathbb{Q}$, $\forall a \in \mathcal{A}_1$. By [3, Proposition 1.11], $(xy)[x, y, z] = y(x[x, y, z])$, let $x \in \mathcal{A}_1$, $y \in M$ and $z = j_0$, then $(xy)^2 \cdot j_0 = \lambda_{(xy)^2} j_0 = 0$. Since $[xy, xy, j_0] = 0$, we have that $(\lambda_{xy})^2 = \lambda_{(xy)^2} = 0$. Hence $\lambda_{xy} = 0$ and thus $xy \in M$. Similarly we prove that $yx \in M$. Using this, it is easily seen that associators belong to M and hence M is an ideal. It follows that either $M = \mathcal{A}_1$ or $M = (0)$ and hence $M = \mathcal{A}_1$.

Observe that if $a \in \mathcal{A}_1$ and $a \cdot j_0 = \lambda j_0$ then $a - \lambda e_1 \in M$. It follows that $\mathcal{A}_1 = M \oplus \mathbb{Q}e_1$. From this and the fact that $M = 0$ if follows that $\mathcal{A}_1$ is one dimensional. Similarly we prove that $\mathcal{A}_N$ is one dimensional.

If $\mathcal{A}_i$ is non-associative, then, by Lemma 3.4 and Proposition 3.2, $\mathcal{A}_i$ has no nilpotent elements and we can write $\mathfrak{A} \cong \oplus_{2 \leq i \leq N-1} \mathcal{A}_i \oplus \mathcal{A}_1 \oplus \mathcal{A}_N \oplus \mathfrak{R}$. By [6, Theorem 3.6, item (iv))] we have $\mathcal{A}_1 \oplus \mathcal{A}_N \oplus \mathfrak{R} \cong T_2(\mathbb{Q})$. □

4 RA-Loops with Hyperbolic Unit Loop $\mathcal{U}(RL)$

In this section we classify the RA-loops L and the ring of algebraic integers $\mathfrak{o}_{\mathbb{K}}$ of a field $\mathbb{K}$, such that the loop of units loop $\mathcal{U}(RL)$ is a hyperbolic loop.We start to look at integral loop rings of finite RA-loops.

Lemma 4.1. *Let L be a finite RA-loop. The loop $\mathcal{U}(\mathbb{Z}L)$ is hyperbolic if, and only if, $\mathcal{U}(\mathbb{Z}L)$ is trivial.*

Proof. As we saw before, there exists a non-abelian finite group G such that $L = M(G, *, g_0) = G\dot{\cup}Gu$. Since $\mathcal{U}(\mathbb{Z}L)$ is hyperbolic we have that $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(\mathbb{Z}G)$. G being finite and [8, Theorem 3.2] implies that $G \in \{S_3, D_4, C_3 \rtimes C_4, C_4 \rtimes C_4\} \cup \{M : M$ is a 2-Hamiltonian group$\}$. By [1, Theorem 3.1], $G' \cong C_2$ and hence $G \notin \{S_3, C_3 \rtimes C_4\}$. We also have that $G \notin \{D_4, C_4 \rtimes C_4\}$, since if this were the case then the algebra $\mathbb{Q}G$ would contain nilpotent elements and thus, by Lemma 3.4, $\mathbb{Q}L$ would contain a copy of Zorn's matrix algebra. Consequently, for some $\mathbb{Z}$-order $\Gamma \subset \mathbb{Q}L$, we would have that $\mathbb{Z}^2 \hookrightarrow \mathcal{U}(\Gamma)$, a contradiction.

Finally, if G is a Hamiltonian 2-group, then, by Proposition 3.3, $\mathcal{U}(\mathbb{Z}L)$ is trivial. □

We can now characterize RA-loops whose integral loop ring has the hyperbolic property.

Theorem 4.2. *Let L be an RA-loop. The loop $\mathcal{U}(\mathbb{Z}L)$ is hyperbolic if, and only if, one of the following holds.*

1. *L is a finite loop*
2. *L is a loop whose center is virtually cyclic. $T(G)$, the torsion subloop of G is abelian with its exponent dividing 4 or 6 and $T(L)$ is a Hamiltonian 2-loop (it can be a group) whose subgroups are all normal in L.*

In either case we have that $\mathcal{U}_1(\mathbb{Z}L) = L$.

Proof. The previous lemma deals with the finite case and so we only have to deal with the case when L is infinite. Obviously we may suppose that L is finitely generated and hence its torsion subloop, $T(L)$, is finite. It is known that the center $\mathcal{Z}(L)$ is a finitely generated abelian group, and so $\mathcal{Z}(L) \cong T(\mathcal{Z}(L)) \times F$, where $T(\mathcal{Z}(L))$ is a finite abelian group and F an abelian torsion free group, (see [1], Lemma 2.1). $\mathcal{U}(\mathbb{Z}L)$ being hyperbolic, gives us that L is hyperbolic and thus F, and hence $\mathcal{Z}(L)$, is virtually cyclic.

Choose an element $z_0 \in \mathcal{Z}(L)$ of infinite order. If $\mathcal{U}(\mathbb{Z}(T(L)))$ is non-trivial then it contains an element y_0 of infinite order and hence $\langle x_0, y_0\rangle$ is a copy of $\mathbb{Z}^2$. Hence we must have that $\mathcal{U}(\mathbb{Z}(T(L)))$ is trivial and, by the previous lemma, $T(L)$ is a hamiltonion 2-loop or 2-group. In particular $\mathbb{Z}T(L)$ does not contain nilpotent elements and hence all subgroups of $T(L)$ are normal in L (this is a standard proof in group rings). So we proved have that L is a finitely generated RA-loop whose torsion subloop $T(L)$ is a Hamiltonian 2-loop and all its subloops are normal in L. Therefore, by [3, Proposition $XII.1.3$], $\mathcal{U}(\mathbb{Z}L) = L[\mathcal{U}(\mathbb{Z}(T(L)))] = LT(L) = L$, i.e., $\mathcal{U}(\mathbb{Z}L)$ is trivial. Since $[L : \mathcal{Z}(L)] = 8$ it follows that the unit group is also virtually cyclic.

□

We now look at the case RL, with $R = \mathfrak{o}_{\mathbb{K}}$ is the ring of algebraic integers of a quadratic extension.

Theorem 4.3. *Let L be a finite RA-loop and let $R = \mathfrak{o}_{\mathbb{K}}$ be the ring of algebraic integers of $\mathbb{K} = \mathbb{Q}(\sqrt{-d}), d \in \mathcal{D}$. The loop of units $\mathcal{U}_1(RL)$ is hyperbolic if, and only if, $L = M_{16}(Q_8)$ and $d \in \mathbb{Z}^+$ with $d \equiv 7 \pmod 8$.*

Proof. Clearly $\mathbb{Z}L \subset \mathfrak{o}_{\mathbb{K}}L$ and thus $\mathcal{U}_1(\mathbb{Z}L)$ is also hyperbolic. By the Lemma 4.1, $\mathcal{U}(\mathbb{Z}L)$ is trivial and $L \cong M_{16}(Q_8) \times E \times A$. Since the hyperbolic loop $\mathcal{U}(\mathfrak{o}_{\mathbb{K}}L) \supset \mathcal{U}(\mathfrak{o}_{\mathbb{K}}Q_8)$, we have that $\mathcal{U}(\mathfrak{o}_{\mathbb{K}}Q_8)$ is a hyperbolic group. Therefore $d \equiv 7 \pmod 8$ and $\mathbb{K}$ is an imaginary extension of $\mathbb{Q}$ (see [9, Theorem 4.7]). It follows that $E = A = 1$.

Conversely, it is well known that $\mathbb{K}L = \mathbb{K}(M_{16}(Q_8)) \cong 8 \cdot \mathbb{K} \oplus (\mathbb{K}, -1, -1, -1)$ ([3, Corollary $VII.2.3$]). By Lemma 2.4, $(\mathbb{K}, -1, -1, -1)$ has the hyperbolic property. Since orders in $\mathbb{K}L$ have commensurable unit loops, we have that $\mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}L)$ is hyperbolic. □

Theorem 4.4. *Let L be an RA-loop and $\mathfrak{o}_{\mathbb{K}}$ be the ring of algebraic integers of $\mathbb{K} = \mathbb{Q}(\sqrt{-d}), d \in \mathcal{D}$. The loop of units $\mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}L)$ is hyperbolic if, and only if, L and d are as follows:*

1. *$L = M_{16}(Q_8)$ and $d \equiv 7 \pmod 8$, $d > 0$.*
2. *L is an infinite virtually cyclic loop whose torsion subloop are all normal. Furthermore, $T(L)$ is an abelian group of exponent dividing 2, if $d > 0$, 4 if $d = 1$ and 6 if $d = 3$.*

In each case we have that $\mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}L) = L$.

Proof. The finite case is settled by the previous theorem and so me may suppose that L is infinite and finitely generated. In particular its torsion subloop is finite.

Since $\mathbb{Z}L \subset \mathfrak{o}_{\mathbb{K}}L$, we have that $\mathcal{U}_1(\mathbb{Z}L)$ is hyperbolic and hence, by Theorem 4.2, $T(L)$ is either a Hamiltonian Moufang 2-loop or an abelian group of exponent dividing 4 or 6, L is virtually cyclic and has a central trivial unit z_0 of infinite order.

If $T(L)$ is a loop, then $\mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}T(L))$ is hyperbolic. By Theorem 4.2, $T(L) = M_{16}(Q_8)$ and since $\mathbf{H}(\mathfrak{o}_{\mathbb{K}})$ has non-trivial units of infinite order it follows that there exists $u_0 \in \mathcal{U}(\mathfrak{o}_{\mathbb{K}}T(L))$ of infinite order (see also [9, Theorem 5.4] or [12, Theorem 1.8.6]). Hence $\langle z_0, u_0\rangle \cong \mathbb{Z}^2$, a contradiction and therefore $T(L)$ is a group.

Theorem 4.2 guarantees that $T(L)$ is an abelian group of exponent dividing 4 or 6. By hypothesis and choice of z_0 we have that $\mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}T(L))$ is trivial. In particular we have that $T(L)$ is an elementary abelian 2-group and $d > 0$, or $T(L)$ is an abelian group of exponent dividing 4 and $d = 1$, or $T(L)$ is an abelian group of exponent dividing 6 and $d = 3$ (see [9, Theorem 3.7]).

Conversely, if $T(L)$ is one of the groups of item (2), then $\mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}T(L))$ is trivial ([9, Theorem 3.7]). As we already observed in Theorem 4.2 we must have that $\mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}L) = L$ and hence it is hyperbolic. $\square$

In the proof of the previous theorem, we claimed the existence of a unit $u \in \mathcal{U}_1(\mathfrak{o}_{\mathbb{K}}M_{16}(Q_8))$ of infinite order which is given by [9, Theorem 5.4]. In fact, let $\epsilon := x + y\sqrt{d}$ be the fundamental invertible of $\mathfrak{o}_{\mathbb{K}}$. We provide two explicit examples:

1. Taking $d = 7$ we have that $\epsilon = 8 + 3\sqrt{7}$. Take $u = 24\sqrt{-7} - (24\sqrt{-7})i - 63j + 64k$; then u is a unit of infinite order and of augmentation 1 (see ([12], Proposition 1.8.2);

2. For $d = 39$ we have that $\epsilon = 25 + 4\sqrt{39}$ and $v = 2\sqrt{-39} - (2\sqrt{-39})i - 12j + 13k$ has infinite order.

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References

[1] de Barros, L. G. X., Juriaans, S. O. *Units in Alternative Integral Loop Rings*, Result. Math., 31 (1997), 266-281.

[2] Bridson, M. R., Haefliger, A. *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.

[3] Goodaire, E. G., Jespers, E., Polcino Milies, C. *Alternative Loop Rings*, Elsevier, Oxford, 1996.

[4] Goodaire, E. G., Polcino Milies, F. C. *When is a unit loop f-unitary?*, Proccedings of the Edinburgh Mathematical Society (2005) 48, 125-142.

[5] Gromov, M. *Hyperbolic Groups, in Essays in Group Theory*, M. S. R. I. publ. 8, Springer, 1987, 75-263.

[6] Iwaki, E., Juriaans, S. O., Souza Filho, A. C. *Hyperbolicity of Semigroup Algebras*, Journal of Algebra, 319(12)(2008), $5000 - 50015$.

[7] Iwaki, E. Jespers, E., Juriaans, S. O., Souza Filho, A. C. *Hyperbolicity of Semigroup Algebras II*, Journal of Algebra and Its Applications, to appear.

[8] Juriaans, S. O., Passi, I. B. S., Prasad, D. *Hyperbolic Unit Groups*, Proceedings of the American Mathematical Society, vol 133(2), 2005, pages 415-423.

[9] Juriaans, S. O., Passi, I. B. S., Souza Filho, A. C. *Hyperbolic $\mathbb{Z}$-Orders and Quaternion Algebras*, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 1, February 2009, pp. $1-14$.

[10] Schafer, R. D.*An Introduction to Nonassociative Algebras*, Academic Press Inc., USA, 1966.

[11] Sehgal, S. K. *Units in Integral Group Rings*, Longman, Harlon, 1994.

[12] Souza Filho, A.C. *Sobre uma Classificação dos Anéis de Inteiros, dos Semigrupos Finitos e dos RA-Loops com a Propriedade Hiperbólica (On a Classification of the Integral Rings, Finite Semigroups and RA-Loops with the Hyperbolic Property)*, PhD. Thesis, IME-USP, São Paulo, 2006, 108 pages.

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